Local transfer and spectra of a diffusive field advected by large-scale incompressible flows

Chuong V. Tran
School of Mathematics and Statistics, University of St Andrews, St Andrews KY16 9SS, United Kingdom
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This study revisits the problem of advective transfer and spectra of a diffusive scalar field in large-scale incompressible flows in the presence of a (large-scale) source. By "large scale" it is meant that the spectral support of the flows is confined to the wave-number region $k<k_d$, where k_d is relatively small compared with the diffusion wave number k_κ. Such flows mediate couplings between neighboring wave numbers within k_d of each other only. It is found that the spectral rate of transport (flux) of scalar variance across a high wave number $k>k_d$ is bounded from above by $Uk_dk\Theta(k,t)$, where U denotes the maximum fluid velocity and $\Theta(k,t)$ is the spectrum of the scalar variance, defined as its average over the shell $(k-k_d,k+k_d)$. For a given flux, say $\vartheta>0$, across $k>k_d$, this bound requires $\Theta(k,t)\geqslant(\vartheta/Uk_d)k^{-1}$. This is consistent with recent numerical studies and with Batchelor's theory that predicts a k^{-1} spectrum (with a slightly different proportionality constant) for the viscous-convective range, which could be identified with (k_d,k_κ). Thus, Batchelor's formula for the variance spectrum is recovered by the present method in the form of a critical lower bound. The present result applies to a broad range of large-scale advection problems in space dimensions $\geqslant 2$, including some filter models of turbulence, for which the turbulent velocity field is advected by a smoothed version of itself. For this case, $\Theta(k,t)$ and ϑ are the kinetic energy spectrum and flux, respectively.

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I. INTRODUCTION

The problem of scalar transport and mixing in turbulent fluid flows has been a subject of active research for decades, dating back to the late 1940s. Early studies by Obukhov [1] and Corrsin [2] applied Kolmogorov's theory of turbulence in a straightforward manner. They found that the scalar (fluid temperature in their case) variance behaved in the same manner as the turbulent kinetic energy, cascading via a $k^{-5/3}$ range to a diffusion range at high wave numbers k for disposal. This result is supposed to apply to cases of relatively small diffusivity κ and viscosity ν in the regime $\kappa\approx\nu$, for which the viscous dissipation and diffusion ranges coincide. Batchelor [3] considered turbulent flows at moderate Reynolds numbers in the regime of large Prandtl or Schmidt number $P_r=\nu/\kappa\gg 1$, for which there exists a broad viscous-convective range $k_\nu\ll k\ll k_\kappa$ between the viscous dissipation wave number k_ν and diffusion wave number k_κ. He found that in this range, the scalar variance spectrum $F(k)$ scales as k^{-1} and is given by

$$F(k)=\frac{\chi}{\gamma}k^{-1}, \tag{1}$$

where χ is the rate at which the scalar variance is dissipated, i.e., the spectral rate of variance transport or variance flux, and γ is an effective least-rate-of-strain parameter given by $\gamma=C(\epsilon/\nu)^{1/2}$. Here ϵ denotes the mean rate of kinetic energy dissipation and C is a constant of order unity. From the Obukhov-Corrsin and Batchelor theories one may visualize a picture of scalar advection in flows at moderate Reynolds numbers in the limit of large P_r, in which a hybrid spectrum obeys the Obukhov-Corrsin $k^{-5/3}$ scaling in the fluid inertial range followed by the Batchelor k^{-1} scaling in the viscous-convective range [4,5]. These pioneering theories have been considered to be breakthroughs and attracted considerable interest to the subject during its infancy [5–9]. Recently, fundamental issues in geophysical, environmental, and industrial applications have sparked a surge in the area, resulting in a huge body of research [10–31] on a variety of dynamical aspects. Another reason for this surge is that computers have become increasingly capable of taking on a scientific problem of this magnitude. Within the past few years, numerical evidence in support of the Batchelor theory and its predicted k^{-1} spectrum has accumulated considerably [4,22,30]. However, this is far from conclusive as the viscous-convective ranges accessible to modern computers are still quite limited. Furthermore, a number of studies [16,24–26] have either argued for or found spectra shallower than the Batchelor spectrum. For these reasons, as well as the phenomenological nature of the Obukhov-Corrsin and Batchelor theories, further theoretical consideration and numerical analysis (with higher resolutions whenever possible) continue to be desirable.

In this study, we revisit the advection-diffusion problem by carrying out a simple but rigorous analysis of the advective transfer term leading to a conclusion that is consistent with the Batchelor picture [3] and with recent numerical results [4,22,30]. We consider large-scale flows, meaning that the tail of the Fourier representation of the flows beyond some finite wave number k_d is identically zero or at least can be ignored. Such smooth flows are relevant for practical purposes as most advection-diffusion problems in the geophysical and environmental contexts are primarily concerned with large-scale advecting flows. They may even model Navier-Stokes turbulence at moderate Reynolds numbers if k_d belongs to the viscous dissipation range and if the exponentially decaying tail of the velocity fields beyond k_d can be ignored. These large-scale flows can mediate transfer of the scalar variance between neighboring wave numbers within k_d of each other only. It is found that the variance flux across a high wave number $k>k_d$ is bounded from above by $Uk_dk\Theta(k,t)$, where U denotes the maximum fluid velocity and $\Theta(k,t)$ is the spectrum of the scalar variance, defined as its average over the shell $(k-k_d,k+k_d)$. From this upper bound, it can be readily deduced that in the high k limit, the flux diminishes if $\Theta(k,t)$ becomes steeper than k^{-1}. Hence, no spectra steeper than k^{-1} could support a nonzero variance flux to the vicinity of the diffusion wave number k_κ in the limit of large k_κ/k_d, which may be identified with P_r (or $P_r^{1/2}$) [31]. Given a persistent source and in the limit $k_\kappa/k_d \to \infty$, this result implies a divergence of the scalar variance not slower than logarithmic in k even if a variance cascade to the small scales is realizable. The present results apply to both passive and active scalars in large-scale flows, provided that in the active case, the excitation of the wave numbers $k > k_d$ of the flows by nonlinear feedback mechanism can be ignored. They also apply to filter models of turbulence, for which the full turbulent velocity is advected by a smoothed version of itself. In this case, $\Theta(k,t)$ is replaced by the kinetic energy spectrum, and the flux in question is the kinetic energy flux.

II. PRELIMINARIES

In this section, we briefly describe the advection-diffusion equations in spectral form, principally to illustrate the coupling locality, which plays a key role in the present analysis. We then recall the variance conservation law and set out a few notations employed in this paper.

The advection-diffusion equations governing the evolution of a diffusive field $\theta(\mathbf{x},t)$ advected by incompressible flows $\mathbf{u}(\mathbf{x},t)$ are

$$\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\theta = \kappa\Delta\theta + f,$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0, \tag{2}$$

where κ is the diffusivity and $f(\mathbf{x},t)$ is a (large-scale) source. The spectral support of $\mathbf{u}(\mathbf{x},t)$ is assumed to be confined to the region $k<k_d$, where k_d is a finite wave number. We consider Eq. (2) in an n-dimensional $(n \geq 2)$ periodic domain, enabling us to express our results conveniently in terms of spatial averages of dynamical quantities. These results can be seen to carry over to an unbounded domain with minimal change. All fields are assumed to have zero spatial average. The advected field $\theta(\mathbf{x},t)$ can be either passive or active. In the latter case, the nonlinear feedback mechanism by $\theta(\mathbf{x},t)$ on $\mathbf{u}(\mathbf{x},t)$ can be arbitrary, as long as it does not "irregularize" $\mathbf{u}(\mathbf{x},t)$ by exciting the small scales of $\mathbf{u}(\mathbf{x},t)$ corresponding to $k>k_d$ to the extent that these scales can no longer be ignored. Furthermore, $\theta(\mathbf{x},t)$ can be a vector, such as the fluid velocity in some filter models of turbulence (for which a pressure term is included).

The Fourier representations of $\mathbf{u}(\mathbf{x},t)$ and $\theta(\mathbf{x},t)$ are

$$\mathbf{u}(\mathbf{x},t) = \sum_{k=|\mathbf{k}|<k_d} \hat{\mathbf{u}}(\mathbf{k},t)\exp\{i\mathbf{k}\cdot\mathbf{x}\} \tag{3}$$

and

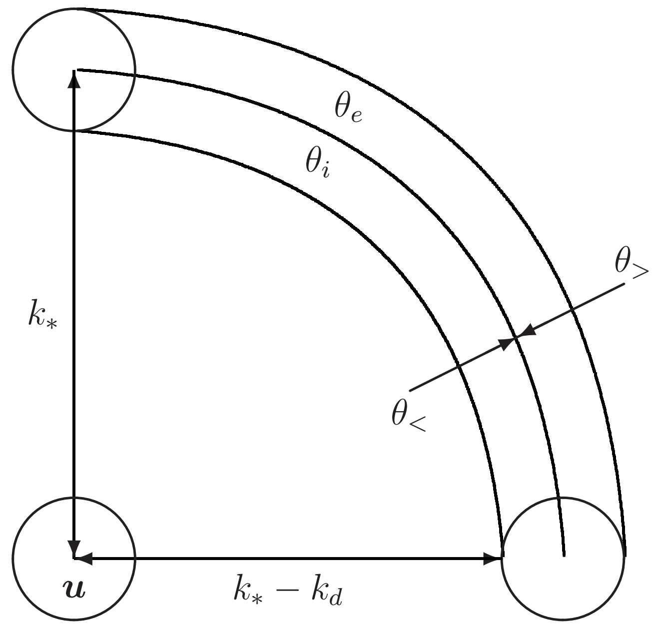

FIG. 1. A schematic description of the spectral supports for $\mathbf{u}$ and for the components $\theta_<$, $\theta_>$, θ_i, and θ_e of θ. The flux term involves only the wave numbers within the shell (k_*-k_d,k_*+k_d), which supports θ_i and θ_e.

$$\theta(\mathbf{x},t) = \sum_{\mathbf{k}} \hat{\theta}(\mathbf{k},t)\exp\{i\mathbf{k}\cdot\mathbf{x}\}, \tag{4}$$

respectively. Here $\mathbf{k}\neq 0$ is the wave vector and $\hat{\mathbf{u}}(\mathbf{k},t)$ and $\hat{\theta}(\mathbf{k},t)$ are the Fourier transforms of $\mathbf{u}(\mathbf{x},t)$ and $\theta(\mathbf{x},t)$, respectively. The reality of $\mathbf{u}(\mathbf{x},t)$ and $\theta(\mathbf{x},t)$ requires $\hat{\mathbf{u}}(\mathbf{k},t) = \hat{\mathbf{u}}^*(-\mathbf{k},t)$ and $\hat{\theta}(\mathbf{k},t)=\hat{\theta}^*(-\mathbf{k},t)$. The incompressibility of $\mathbf{u}(\mathbf{x},t)$ further requires $\mathbf{k}\cdot\hat{\mathbf{u}}(\mathbf{k},t)=0$. In spectral form, the first equation of Eq. (2) becomes

$$\frac{\partial}{\partial t}\hat{\theta}(\mathbf{k},t) = \sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \mathbf{k}'\cdot\hat{\mathbf{u}}(\mathbf{k}'',t)\hat{\theta}(\mathbf{k}',t) - \kappa k^2\hat{\theta}(\mathbf{k},t) + \hat{f}(\mathbf{k},t), \tag{5}$$

where $\hat{f}(\mathbf{k},t)$ is the Fourier transform of $f(\mathbf{x},t)$. The incompressibility of $\mathbf{u}(\mathbf{x},t)$ manifests itself in Eq. (5) through the fact that $\mathbf{k}'\cdot\hat{\mathbf{u}}(\mathbf{k}'',t)\hat{\theta}(\mathbf{k}',t)=0$ if $\mathbf{k}'$ and $\mathbf{k}''$ are collinear. The triad relation $\mathbf{k}=\mathbf{k}'+\mathbf{k}''$, together with the constraint $k''=|\mathbf{k}''|<k_d$, implies that $k'=|\mathbf{k}'|$ satisfies $|k-k'|<k_d$. This means that a given $k>k_d$ can couple with other wave numbers within the shell $(k-k_d,k+k_d)$ only. This coupling locality has a significant consequence as will be seen in the next section.

Given a "reference" wave number k_* $(k_* \geq k_d)$, let us denote by $\theta_<$, $\theta_>$, θ_i, and θ_e the components of θ spectrally supported by the ball $b=\{\mathbf{k}:k\leq k_*\}$, its complement $B=\{\mathbf{k}:k>k_*\}$, the inner shell $S_i=\{\mathbf{k}:k_*-k_d<k\leq k_*\}$, and the outer shell $S_e=\{\mathbf{k}:k_*<k<k_*+k_d\}$, respectively (see Fig. 1). For example,

$$\theta_{>}(\mathbf{x},t) = \sum_{\mathbf{k}\in B} \hat{\theta}(\mathbf{k},t)\exp\{i\mathbf{k}\cdot\mathbf{x}\}. \tag{6}$$

For $k > k_d$, let $\Theta(k,t)$ denote the averaged variance taken over the shell $(k-k_d, k+k_d)$. For example, $\Theta(k_*,t)$ is given by

$$\Theta(k_*,t) = \frac{\langle\theta_i^2\rangle + \langle\theta_e^2\rangle}{2k_d}, \tag{7}$$

where $\langle\cdots\rangle$ denotes a spatial average. It is evident that $\Theta(k,t)$ approximates the usual spectrum $F(k,t)$. In general, the approximation can become increasingly better for higher k as the shell $(k-k_d, k+k_d)$ becomes thinner, in the sense that the ratio of the shell thickness $2k_d$ to its radius k becomes smaller. For power-law spectra, $\Theta(k,t)$ actually approaches $F(k,t)$ in the limit $k/k_d \to \infty$. For example, for the Batchelor spectrum given by Eq. (1), we have

$$\Theta(k_*) = \frac{\chi}{2k_d\gamma}\int_{k_*-k_d}^{k_*+k_d} k^{-1}dk = \frac{\chi}{2k_d\gamma}\ln\frac{k_*+k_d}{k_*-k_d}$$
$$= \frac{\chi}{2k_d\gamma}\ln\left(1+\frac{2k_d}{k_*-k_d}\right), \tag{8}$$

which indeed tends to $\chi(\gamma k_*)^{-1}$ in the limit $k_*/k_d \to \infty$. Thus $\Theta(k,t)$ tends to $F(k,t)$ in the limit of high k (at least for power-law spectra). We will express our results in terms of $\Theta(k,t)$ instead of $F(k,t)$ since the former arises more naturally in the present context.

Given periodic functions ϕ and ψ having zero mean and bounded mean-square gradients, we have

$$\langle\phi\mathbf{u}\cdot\nabla\psi\rangle = -\langle\psi\mathbf{u}\cdot\nabla\phi\rangle. \tag{9}$$

This identity gives rise to a wealth of conservation laws, particularly the variance conservation law, and is used repeatedly in this study.

III. MAIN RESULTS

We now present the main results of this study. First, we elaborate on the locality of the variance transfer and then derive the lower bound for $\Theta(k,t)$ as described earlier. Second, we show that given a persistent source, $\langle\theta^2\rangle$ grows without bound in the limit $k_\kappa \to \infty$, irrespective of the underpinning dynamics. Third, we compare the present finding with a recent result [31] derived on the assumption of bounded velocity gradients, i.e., $|\nabla\mathbf{u}| < \infty$, which is a weaker condition than the present one. Finally, the slight discrepancy between the present finding and Batchelor's formula (1) is discussed.

A. Bounds for variance flux and spectrum

The governing equation for the evolution of the small-scale variance $\langle\theta_{>}^2\rangle$ is obtained by multiplying Eq. (2) by $\theta_{>}$ and taking the spatial average of the resulting equation

$$\frac{1}{2}\frac{d}{dt}\langle\theta_{>}^2\rangle = -\langle\theta_{>}\mathbf{u}\cdot\nabla\theta\rangle - \kappa\langle|\nabla\theta_{>}|^2\rangle$$
$$= -\langle\theta_{>}\mathbf{u}\cdot\nabla\theta_{<}\rangle - \kappa\langle|\nabla\theta_{>}|^2\rangle, \tag{10}$$

where Eq. (9) and the linearity of the advection term have been used and the forcing term vanishes as B is assumed to be source free. The triple-product term (flux term) in Eq. (10) represents the net variance transfer across k_* into the region $k > k_*$, which drives the small-scale dynamics. At the modal level, this flux term consists of triple-product terms of the form $\hat{\theta}(\mathbf{k},t)\hat{\theta}(\mathbf{k}',t)\mathbf{k}'\cdot\hat{\mathbf{u}}(\mathbf{k}'',t)$, where $\mathbf{k}\in B$, $\mathbf{k}'\in b$, and $\mathbf{k} = \mathbf{k}' + \mathbf{k}''$. Since $k'' < k_d$, this triad relation implies that k and k' can couple only if $k - k' < k_d$. Hence, only modes in θ_i and θ_e, i.e., within the wave number shell (k_*-k_d, k_*+k_d), contribute to the flux term (see Fig. 1). For this reason, the variance transfer can be considered as being highly local, particularly at high k, where the shell (k_*-k_d, k_*+k_d) becomes relatively thin (radius becoming larger but thickness remaining fixed). Here, we use the term "highly local" to emphasize the fact that $k/k' \to 1$ in the limit $k_*/k_d \to \infty$. This term is to distinguish the present couplings from those of a lesser degree of locality between $k \approx k' \approx k_*$ via $k'' \approx k_*$, where the ratio k/k' remains strictly greater than unity in the same limit. Such couplings are clearly absent from the flux term. From this geometric consideration, we can write

$$\langle\theta_{>}\mathbf{u}\cdot\nabla\theta_{<}\rangle = \langle\theta_e\mathbf{u}\cdot\nabla\theta_i\rangle. \tag{11}$$

Substituting this result into Eq. (10) yields

$$\frac{1}{2}\frac{d}{dt}\langle\theta_{>}^2\rangle = -\langle\theta_e\mathbf{u}\cdot\nabla\theta_i\rangle - \kappa\langle|\nabla\theta_{>}|^2\rangle$$
$$\leq U\langle|\theta_e||\nabla\theta_i|\rangle - \kappa\langle|\nabla\theta_{>}|^2\rangle$$
$$\leq U\langle\theta_e^2\rangle^{1/2}\langle|\nabla\theta_i|^2\rangle^{1/2} - \kappa\langle|\nabla\theta_{>}|^2\rangle$$
$$\leq Uk_*\langle\theta_e^2\rangle^{1/2}\langle\theta_i^2\rangle^{1/2} - \kappa\langle|\nabla\theta_{>}|^2\rangle$$
$$\leq \frac{Uk_*}{2}(\langle\theta_e^2\rangle + \langle\theta_i^2\rangle) - \kappa\langle|\nabla\theta_{>}|^2\rangle$$
$$= Uk_dk_*\Theta(k_*,t) - \kappa\langle|\nabla\theta_{>}|^2\rangle, \tag{12}$$

where, as we recall, U denotes the maximum fluid velocity and $\Theta(k,t)$ is the variance spectrum defined by Eq. (7). In Eq. (12), the Cauchy-Schwarz inequality and the self-explanatory (Poincaré-type) inequality $\langle|\nabla\theta_i|^2\rangle \leq k_*^2\langle\theta_i^2\rangle$ have been used. The bound for the flux term in Eq. (12) is interesting and can be readily interpreted in what follows.

For a positive flux through k_*, say ϑ_*, the final estimate in Eq. (12) implies that

$$\vartheta_* \leq Uk_dk_*\Theta(k_*,t) \tag{13}$$

or, equivalently,

$$\Theta(k_*,t) \geqslant \frac{\vartheta_*}{Uk_d}k_*^{-1}. \tag{14}$$

It follows that a positive k-independent flux is possible only if $\Theta(k,t)$ becomes no steeper than k^{-1} (pointwise) for high k. This constraint is consistent with Batchelor's theory that predicts a k^{-1} spectrum for the viscous-convective range, which could be identified with (k_d,k_κ). Since Eq. (14) implies a divergence of $\langle\theta^2\rangle$ toward the small scales at least as rapid as logarithmic in k, a positive variance flux to ever smaller scales [including those that diminish no more rapidly than $(\ln k)^{-1}$] requires *a priori* an unbounded variance "passage." This is in a sharp contrast to the classical direct energy cascade (and the Obukhov-Corrsin variance cascade), which is supposed to proceed through an inertial range virtually free of energy. In some sense, the energy cascade is rather "rushing," whereas the variance cascade of the present case (if realizable) would be far less dramatic, "leaking" through a fully filled inertial range. In the presence of a persistent scalar source, $\langle\theta^2\rangle$ necessarily grows without bound in the limit $k_\kappa\to\infty$, for obvious reasons. On the one hand, a variance cascade to ever-smaller scales already requires at least a logarithmic divergence of $\langle\theta^2\rangle$ toward the small scales. On the other hand, if such a cascade is unrealizable, the injected variance is necessarily trapped at the large scales, thereby resulting in their unbounded growth. For the sake of completeness, this argument will be made more quantitative in the next subsection.

B. Unbounded variance growth in the limit $k_\kappa\to\infty$

Similar to Eq. (10), the governing equation for the evolution of the large-scale variance $\langle\theta_<^2\rangle$ is obtained by multiplying Eq. (2) by $\theta_<$ and taking the spatial average of the resulting equation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle\theta_<^2\rangle &= -\langle\theta_<\mathbf{u}\cdot\nabla\theta\rangle-\kappa\langle|\nabla\theta_<|^2\rangle+\vartheta\\ &=\langle\theta_>\mathbf{u}\cdot\nabla\theta_<\rangle-\kappa\langle|\nabla\theta_<|^2\rangle+\vartheta\\ &=\langle\theta_e\mathbf{u}\cdot\nabla\theta_i\rangle-\kappa\langle|\nabla\theta_<|^2\rangle+\vartheta\\ &\geqslant -Uk_dk_*\Theta(k_*,t)-\kappa\langle|\nabla\theta_<|^2\rangle+\vartheta,\end{aligned} \tag{15}$$

where Eqs. (9) and (11) have been used and the inequality is a straightforward application of the upper bound for the flux term derived in Eq. (12). In Eq. (15), $\vartheta=\langle\theta_<f\rangle$ is the scalar variance injection rate. For some large time $t=T$, say $T=1/(2\kappa k_*^2)$, let $\bar{Q}$ denote the average over $[0,T]$ of a dynamical quantity Q. Taking the time average of Eq. (15) and rearranging the terms in the resulting equation yields

$$\kappa k_*^2\langle\theta_<^2\rangle+\kappa\overline{\langle|\nabla\theta_<|^2\rangle}\geqslant\bar{\vartheta}-Uk_dk_*\bar{\Theta}(k_*), \tag{16}$$

where the initial value of $\langle\theta_<^2\rangle$ has been omitted for convenience. Upon making the substitution $\langle|\nabla\theta_<|^2\rangle\leqslant k_*^2\langle\theta_<^2\rangle$ in Eq. (16), we obtain

$$\kappa k_*^2\langle\theta_<^2\rangle+\kappa k_*^2\overline{\langle\theta_<^2\rangle}\geqslant\bar{\vartheta}-Uk_dk_*\bar{\Theta}(k_*). \tag{17}$$

In accord with a persistent source, let us assume $\bar{\vartheta}>0$. Now in the limit $k_\kappa\to\infty$ $(\kappa\to 0)$, if there exists no $k_*<\infty$ such that the right-hand side of Eq. (17) is positive, then $\langle\theta^2\rangle$ diverges toward the small scales as discussed above. On the other hand, if there exists $k_*<\infty$ such that the right-hand side of Eq. (17) is positive, then $(\langle\theta_<^2\rangle+\overline{\langle\theta_<^2\rangle})\to\infty$. It follows that $\langle\theta_<^2\rangle\to\infty$, and hence $\langle\theta^2\rangle\to\infty$. Thus, $\langle\theta^2\rangle$ diverges regardless of whether or not there is a variance cascade.

C. Discussion

When $\mathbf{u}(\mathbf{x},t)$ is not restricted to the large scales, there are no constraints on k' and k in the triple-product terms $\hat{\theta}(\mathbf{k},t)\hat{\theta}(\mathbf{k}',t)\mathbf{k}'\cdot\hat{\mathbf{u}}(\mathbf{k}'',t)$ contributing to $\langle\theta_>\mathbf{u}\cdot\nabla\theta_<\rangle$. The flux term then involves, in principle, couplings for every $k'\leqslant k_*$ and $k>k_*$. The presence of nonlocal couplings (between $k'\ll k_*$ and $k\gg k_*$ via $k''\approx k$) and the other type of local couplings (between $k\approx k'\approx k_*$ via $k''\approx k_*$) mentioned earlier effectively makes the flux term unmanageable by the present method, in the sense that its analytic estimates would be too excessive for meaningful interpretations. For this case, Tran [31] finds by examining the evolution equation for the scalar gradients that if the advecting velocity fields have bounded gradients, then diffusion anomaly, i.e., a variance cascade to ever smaller scales, requires the variance at the small scales to be no less than that provided by the Batchelor k^{-1} spectrum. This constraint is weaker than the present one as it does not rule out the possibility of bounded variance corresponding to non-power-law spectra having gaps of severe variance deficiency in the intermediate wave-number region, provided that the variance requirement at the small scales is met. The present finding, by exploiting the high locality of the variance transfer for large-scale advecting flows, rules out this possibility. The variance is required to grow without bound either via bounded spectra not steeper than k^{-1} (pointwise) if a variance cascade is realizable or via unbounded spectra if otherwise.

In the absence of a scalar source, a finite variance reservoir cannot support a k-independent flux because such a flux requires an unbounded variance "passage" as we have concluded. Our result allows for no significant "chunk" of a given initial variance reservoir $\langle\theta_0^2\rangle$ at large scales to break away and cascade to the small scales by itself. Rather, it suggests a gradual spread out of $\langle\theta_0^2\rangle$ ever more thinly in wave-number space, giving rise to a diminishing flux, which can be readily estimated. Suppose that at a later time, a k^{-1} range gets established from k_d to $k_*\gg k_d$ or beyond. Then, in this range, the spectrum $\Theta(k,t)$ is bounded by $\Theta(k,t)\leqslant\langle\theta_0^2\rangle k^{-1}/\ln(k_*/k_d)$. Upon substituting this into Eq. (13), we obtain

$$\vartheta_*\leqslant\frac{Uk_d\langle\theta_0^2\rangle}{\ln(k_*/k_d)}. \tag{18}$$

This means that ϑ_* diminishes at least as rapidly as $[\ln(k_*/k_d)]^{-1}$. Note that although a logarithmic decay of the flux can be expected on heuristic grounds, Eq. (18) may not be rigorously derived without the constraint (13).

The present bound (14) for $\Theta(k,t)$ resembles the Batchelor formula (1) in every aspect except that Uk_d in Eq. (14) plays the role of γ in Eq. (1). This apparent discrepancy, however, can be reconciled if we reformulate the present problem in accord with the Batchelor setting. It can be seen that the product Uk_d is essentially an upper bound for the velocity gradients $|\nabla \mathbf{u}|$. So if we identify $\nu(Uk_d)^2 \approx \nu|\nabla \mathbf{u}|^2$ with the kinetic energy dissipation rate ϵ in the Batchelor setting of turbulent advection, then we obtain $Uk_d \approx (\epsilon/\nu)^{1/2} \approx \gamma$. Hence, Eqs. (1) and (14) agree. This is no surprise because the Batchelor problem would reduce to the present case upon the hypothesis that the exponentially decaying tail (beyond k_d) of the turbulent velocity field contributes negligibly to the advective variance transfer. As it stands, Eq. (14) captures the intuitive physical fact that for fixed U, flows at larger scales (smaller k_d) are poorer transporters as scalar spectra having larger spectral amplitudes, i.e., larger factors $\vartheta_*/(Uk_d)$, would be required to support the variance flux ϑ_* across k_*.

IV. TURBULENT ENERGY TRANSFER BY LARGE-SCALE ADVECTION

The above results apply to the energy transfer by large-scale advection in turbulence. Namely, the advection of the turbulent velocity by its large-scale component alone results in a contributing energy flux that vanishes at high k if the energy spectrum becomes steeper than k^{-1}. For the Kolmogorov $k^{-5/3}$ spectrum, this means that the large-scale advection contributes negligibly to the direct energy transfer. On physical grounds, this is consistent with the expectation that the large scales, while advecting the turbulent eddies, do not stretch them significantly. The remaining of this paper is devoted to detailed elaboration of this fact.

We begin by recalling the Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \nu\Delta\mathbf{v} + \mathbf{f},$$

$$\nabla\cdot\mathbf{v} = 0, \tag{19}$$

where $\mathbf{v}(\mathbf{x},t)$ is the fluid velocity, $p(\mathbf{x},t)$ is the pressure, and $\mathbf{f}(\mathbf{x},t)$ is a large-scale forcing. Let $\mathbf{u}$ be a large-scale component of $\mathbf{v}$, as defined by Eq. (3), and $\mathbf{u}'$ be its small-scale complement, i.e., $\mathbf{v}=\mathbf{u}+\mathbf{u}'$. Furthermore, let $\mathbf{v}_<$, $\mathbf{v}_>$, $\mathbf{v}_i$, $\mathbf{v}_e$, and $V(k)$ be defined in the same ways as $\theta_<$, $\theta_>$, θ_i, θ_e, and $\Theta(k)$, respectively. Note that $V(k)$ is approximately twice the usual kinetic energy spectrum and that all the components of $\mathbf{v}$ so defined are incompressible. Similar to Eq. (12), the evolution of the small-scale energy $\langle|\mathbf{v}_>|^2\rangle/2$ is governed by

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle|\mathbf{v}_>|^2\rangle &= -\langle\mathbf{v}_>\cdot(\mathbf{v}\cdot\nabla)\mathbf{v}_<\rangle - \nu\langle|\nabla\mathbf{v}_>|^2\rangle\\ &= -\langle\mathbf{v}_>\cdot(\mathbf{u}\cdot\nabla)\mathbf{v}_<\rangle - \langle\mathbf{v}_>\cdot(\mathbf{u}'\cdot\nabla)\mathbf{v}_<\rangle\\ &\quad - \nu\langle|\nabla\mathbf{v}_>|^2\rangle\\ &= -\langle\mathbf{v}_e\cdot(\mathbf{u}\cdot\nabla)\mathbf{v}_i\rangle - \langle\mathbf{v}_>\cdot(\mathbf{u}'\cdot\nabla)\mathbf{v}_<\rangle - \nu\langle|\nabla\mathbf{v}_>|^2\rangle\\ &\leqslant Uk_dk_*V(k_*,t) - \langle\mathbf{v}_>\cdot(\mathbf{u}'\cdot\nabla)\mathbf{v}_<\rangle - \nu\langle|\nabla\mathbf{v}_>|^2\rangle,\end{aligned} \tag{20}$$

where the forcing and pressure terms vanish as the region under consideration is assumed to be force free and $\mathbf{v}_>$ is incompressible. In the final equation of Eq. (20), the first term on the right-hand side represents an upper bound for the energy transfer across k_* due to large-scale advection and the second term is the energy transfer across k_* due to small-scale advection. The former vanishes for high k_* if $V(k,t)$ becomes steeper than k^{-1}. This means that the latter is solely responsible for the direct energy cascade in the classical picture of turbulence, for which the $k^{-5/3}$ energy inertial range is far too steep for the former to make a non-negligible contribution. At the modal level, this result is consistent with the expectation that triad interactions involving well-separated scales [those due to large-scale advection in the flux term $\langle\mathbf{v}_e\cdot(\mathbf{u}\cdot\nabla)\mathbf{v}_i\rangle$] are relatively weak. Note that not all triads of well-separated scales are contained within $\langle\mathbf{v}_e\cdot(\mathbf{u}\cdot\nabla)\mathbf{v}_i\rangle$ as the term $\langle\mathbf{v}_>\cdot(\mathbf{u}'\cdot\nabla)\mathbf{v}_<\rangle$ also has this type of triads. Such triads are formed by large-scale modes in $\mathbf{v}_<$ and small-scale modes in $\mathbf{v}_>$ and $\mathbf{u}'$. As their counterparts in $\langle\mathbf{v}_e\cdot(\mathbf{u}\cdot\nabla)\mathbf{v}_i\rangle$, these can be shown to be relatively weak and not responsible for the classical direct energy cascade.

The above result may be applicable to models of turbulence that are derived by regularizing the Navier-Stokes equations by a variety of filtering techniques (see Graham *et al.* [32] for a discussion of several such models). For example, let us consider the "Leray" model, obtained by dropping the small-scale component $\mathbf{u}'$ from the advecting velocity in the Navier-Stokes system, i.e.,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{v} + \nabla p = \nu\Delta\mathbf{v} + \mathbf{f},$$

$$\nabla\cdot\mathbf{v} = 0. \tag{21}$$

For this simple model, the governing equation for $\langle|\mathbf{v}_>|^2\rangle/2$ is given by Eq. (20) without the small-scale advection term $\langle\mathbf{v}_>\cdot(\mathbf{u}'\cdot\nabla)\mathbf{v}_<\rangle$. As a consequence, the classical direct energy cascade is not realizable for the reason discussed in the preceding paragraph. Instead, the energy behaves in the same manner as the variance $\langle\theta^2\rangle$ described earlier. Namely, the energy either cascades to the small scales via spectra not steeper than k^{-1} or else accumulates at the large scales. Given a persistent source of energy, i.e., $\overline{\langle\mathbf{v}\cdot\mathbf{f}\rangle}>0$, the energy necessarily grows without bound in the inviscid limit. Equation (21) represents a class of regularization models of turbulence, which have been studied widely as alternatives to subgrid-scale models [32] and for which the k^{-1} scaling for the energy spectrum has been found by phenomenological arguments. The present result provides a different perspective to this possible scaling.

In passing, we would like to note that the question of realizability of a (variance or energy) cascade and the associated k^{-1} (or shallower) spectrum cannot be resolved by the present analysis. This question is challenging because a lower bound for the flux term is highly infeasible, even for very simple flows. Given this difficulty, one may be better off resorting to numerical methods. What we have shown here is that if there is a cascade, then it must proceed through spectra not steeper than k^{-1} (pointwise). The critical k^{-1} scaling can be seen as most plausible for a number of reasons. In particular, it would correspond to a cascade of maximal spectral extent.

V. CONCLUDING REMARKS

In summary, we have examined the advective transfer and spectral scaling of a diffusive field $\theta(\mathbf{x},t)$ in large-scale incompressible flows $\mathbf{u}(\mathbf{x},t)$, whose spectral support is confined to the wave-number region $k<k_d$, for some finite wave number k_d, which is relatively small compared with the diffusion wave number k_κ. The main result obtained is the upper bound $Uk_dk_*\Theta(k_*,t)$ for the variance flux across a high wave number $k_*>k_d$. Here U denotes the maximum fluid velocity and $\Theta(k,t)$ is the variance spectrum, defined as its average over the shell $(k-k_d,k+k_d)$. The derivation of this bound exploits the very fact that the advecting flows under consideration mediate variance transfer between neighboring wave numbers within k_d of each other only. The derived bound implies that for $k \gg k_d$, a nonzero k-independent flux is possible only if $\Theta(k,t)$ becomes no steeper than k^{-1} (pointwise). This result is consistent with Batchelor's theory and with recent numerical and theoretical results [4,22,30,31]. One element of the present findings is the pointwise constraint on $\Theta(k,t)$ in Eq. (14). Given this constraint and a persistent source, the variance is required to grow without bound in the limit $k_\kappa/k_d \to \infty$ $(\kappa \to 0)$, irrespective of the underpinning dynamics. The present results have been shown to apply to the Leray model of turbulence, for which the turbulent velocity is advected by a smoothed version of itself. Furthermore, they apply equally to different space dimensions $\geqslant 2$ as the analysis is dimension independent. Finally, the possible relation between the advecting and advected fields does not enter the calculations. Hence, the results are valid for both passive and active scalars, provided that in the latter case, the nonlinear feedback mechanisms on the flows do not alter their large-scale designation.

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